1. (a) [5] We place 9 distinct books from left to right on a shelf. Find the number of possible arrangements in which books A and B are **not** adjacent to each other.

   (b) [5] A machine produces 6-sided dice. The machine is defective: while 99.9% of the dice it produces are normal, the remaining 0.1% have all their faces marked 6. Suppose I take (at random) a die produced by this machine and roll it n times, and then I inform you that all the rolls resulted in 6. For which values of n do you now think it is more likely that I took a defective die than that I took a normal die?

2. (a) [7] Let $X_1, \ldots, X_n$ be independent $\mathcal{N}(0,1)$ random variables. Let

$$Y_1 = \left|\frac{1}{n}\sum_{i=1}^{n} X_i\right|, \quad Y_2 = \frac{1}{n}\sum_{i=1}^{n} |X_i|.$$

   Find $\mathbb{E}(Y_1)$ and $\mathbb{E}(Y_2)$.

   (b) [7] Assume X follows the exponential distribution with parameter 1. Find $\mathbb{E}(\lfloor X \rfloor)$.

3. In a field, hunters attempt to shoot ducks which are flying by. Assume that there are h hunters and d ducks, each hunter chooses a duck uniformly at random and shoots, hitting (and killing) it with probability $p \in (0,1)$, and the choices and shots of all hunters are independent. Let D be the number of ducks which die.

   (a) [5] Find $\mathbb{E}(D)$.

   (b) [4] Let A be the event that each duck is chosen by at most one hunter. Show that, when h and p are kept fixed and d tends to infinity, $\mathbb{P}(A)$ tends to 1.

   (c) [3] Prove that, when h and p are kept fixed and d tends to infinity, D converges in distribution to a random variable X; identify the distribution of X.
   Hint. For $m \in \{0, \ldots, h\}$, write: $\mathbb{P}(D = m) = \mathbb{P}(\{D = m\} \mid A) \cdot \mathbb{P}(A) + \mathbb{P}(\{D = m\} \mid A^c) \cdot \mathbb{P}(A^c)$.

4. Let X be a random variable following the Poisson(λ) distribution. Let Y be a random variable such that, for each $n \in \{0, 1, 2, \ldots\}$, the probability that Y equals n is proportional to $n^2 \cdot f_X(n)$.

   (a) [7] Find the probability mass function of Y.

   (b) [7] Find the moment-generating function of Y.

5. Let X and Y be independent random variables following the exponential distribution with parameter 1.

   (a) [7] Let $Z = X + 2Y$. Find $f_{Z,Y}$, the joint probability density function of Z and Y.

   (b) [7] Let U be the smallest of the two values X, Y (that is, $U = \min(X,Y)$) and V be the largest of the two values X, Y (that is, $V = \max(X,Y)$). Find $F_{U,V}$, the joint cumulative distribution function of U and V.

6. (a) [7] Let $X_1, \ldots, X_n$ be independent Bernoulli(p) random variables. Prove, using moment-generating functions, that $\sum_{i=1}^n X_i$ follows the Binomial(n,p) distribution.

   (b) [7] For $p \in (0,1)$, $\varepsilon > 0$ and $n \in \mathbb{N}$, let $I(p, \varepsilon, n)$ be the set

$$\{i \in \mathbb{N} : (1-\varepsilon)np \le i \le (1+\varepsilon)np\}.$$

   Prove that, for fixed p and ε,

$$\lim_{n\to\infty} \sum_{i \in I(p,\varepsilon,n)} \binom{n}{i} p^i (1-p)^{n-i} = 1.$$

7. In an exam, students' scores are between 0 and 100. Assume that the score that any individual student obtains in the exam is a random variable with mean 70 and standard deviation 9, and that scores of distinct students are independent. An instructor gives the exam to two classes, one of 55 students and another of 90 students. Estimate the probability that the average score in the class with 90 students exceeds that of the other class by at least 2 points.
   If your calculator doesn't find square roots, you may use the approximations: $\sqrt{2} \approx 1.41$, $\sqrt{5} \approx 2.24$, $\sqrt{11} \approx 3.32$, $\sqrt{29} \approx 5.39$.

Standard Normal cumulative distribution function
The value given in the table is $F_X(x)$ for $X \sim \mathcal{N}(0,1)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000